Dynamics Concentration of Large-Scale Tightly-Connected Networks

Hancheng Min and Enrique Mallada

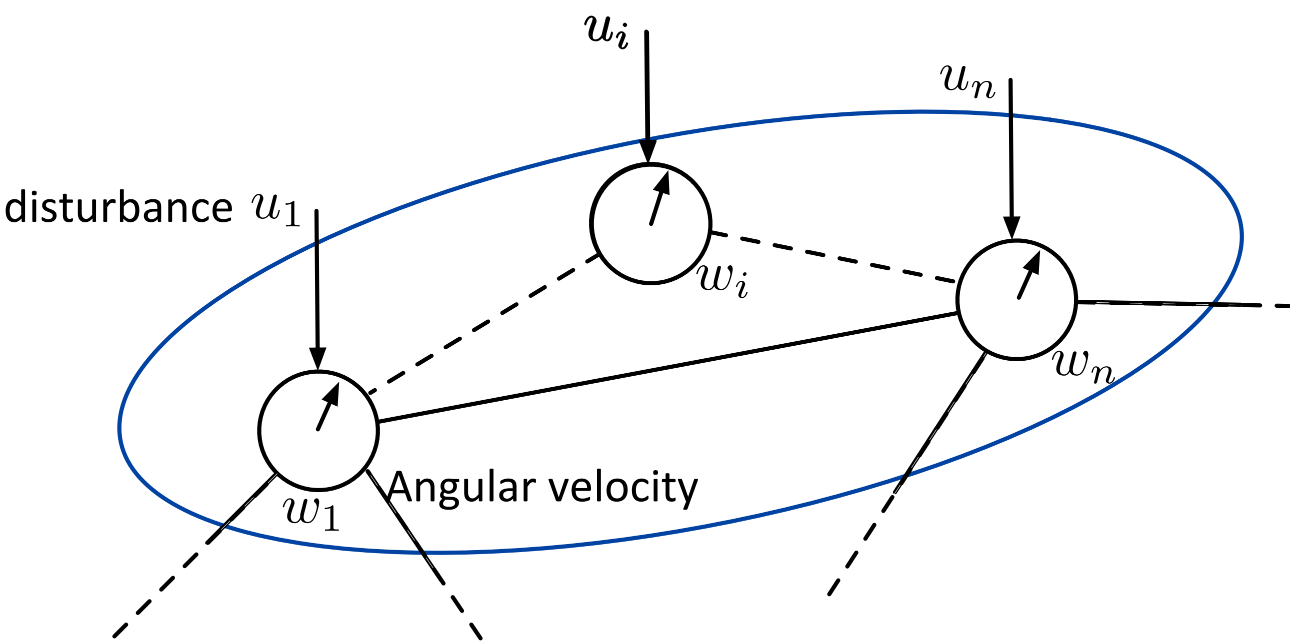
JOHNS HOPKINS UNIVERSITY

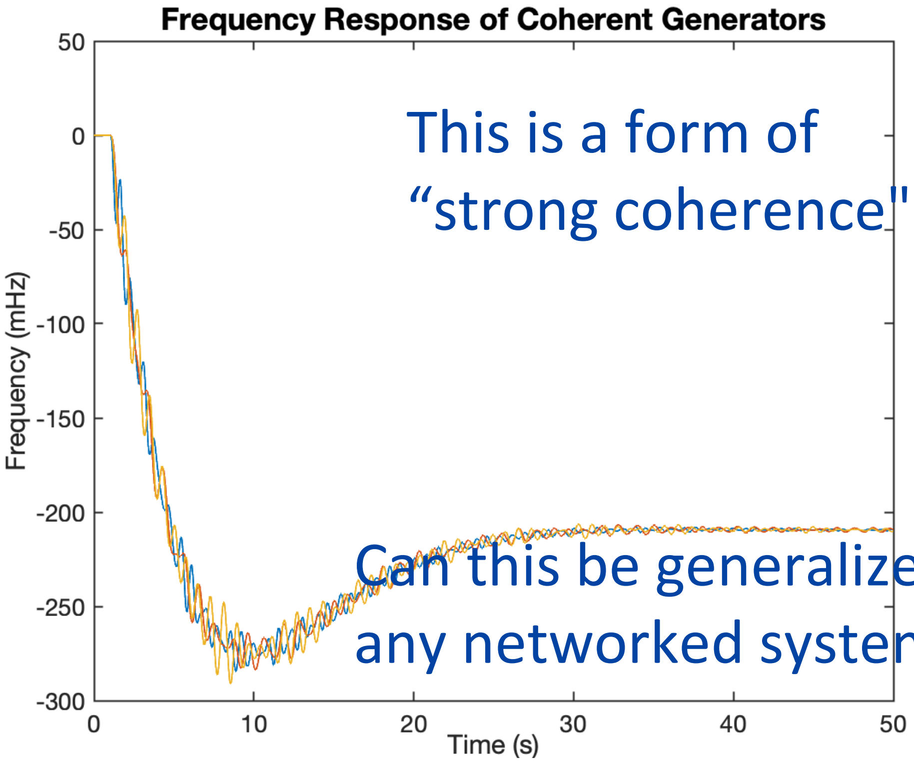
58th Conference on Decision and Control

December 11-13, 2019

Examples of Coherence: Synchronous Generators

In power grids, a group of synchronous generators is coherent if they have similar frequency responses.





This is a form of "strong coherence"

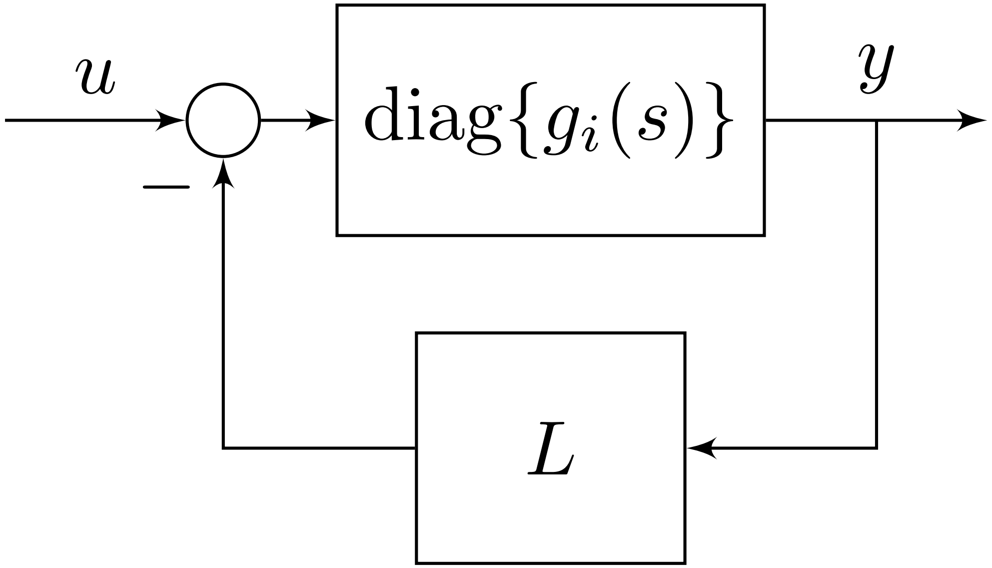
Can this be generalized to any networked systems?

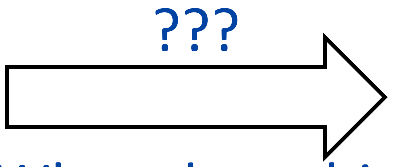
This Talk

- Characterization of **coherent dynamics**
- **Dynamics concentration** of large-scale tightly-connected network
- Numerical illustrations

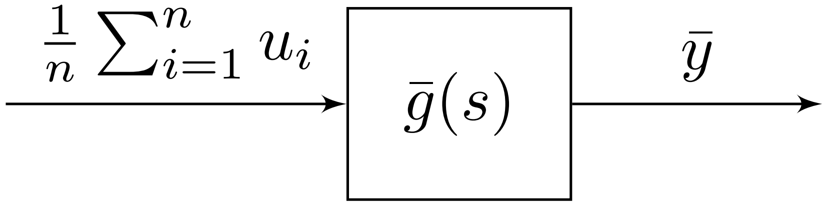
Coherence in networked dynamical systems

Block Diagram:





When does this network exhibits strong coherence?



What is the **coherent dynamics** of the network?

$g_i(s), i = 1, \cdots, n,$ Node Dynamics,

L symmetric real Laplacian matrix,

$0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L),$

$L = V\Lambda V^T, \ \Lambda = \mathrm{diag}\{\lambda_1(L), \lambda_2(L), \cdots, \lambda_n(L)\}$

Coherence in networked dynamical systems: Homogenous case

The transfer matrix from input **u** to output **y** :

$$
\begin{aligned}
T(s) &= (I_n + \mathrm{diag}\{g_i(s)\}L)^{-1}\mathrm{diag}\{g_i(s)\} \\
&= (\mathrm{diag}\{g_i^{-1}(s)\} + L)^{-1} \\
&= (\mathrm{diag}\{g_i^{-1}(s)\} + V\Lambda V^T)^{-1} \\
&= V(V^T\mathrm{diag}\{g_i^{-1}(s)\}V + \Lambda)^{-1}V^T
\end{aligned}
$$

$$V = [\mathbb{1}, V_\perp]$$

$$\mathbb{1} = [1, \cdots, 1]^T$$

Assume homogeneity: $g_i(s) = g(s), \ i = 1, \cdots, n$

$$T(s) = \boxed{\frac{1}{n}g(s)\mathbb{1}\mathbb{1}^T} + \boxed{V_\perp \mathrm{diag}\left\{\frac{1}{g^{-1}(s) + \lambda_i(L)}\right\}_{i=2}^{n} V_\perp^T}$$

Coherent dynamics independent of the network structure

Dynamics depending on the network structure

Coherence in networked dynamical systems: Homogenous case

For any s_0,

$$T(s_0) = \frac{1}{n} g(s_0) \mathbb{1}\mathbb{1}^T + \boxed{V_\perp \operatorname{diag}\left\{ \frac{1}{g^{-1}(s_0) + \lambda_i(L)} \right\}_{i=2}^{n} V_\perp^T}$$

The 2-norm converges to 0 as $\lambda_2(L)$ increases

Therefore, for any s_0 which is not a pole of g(s), we have

$$\lim_{\lambda_2(L) \to +\infty} \left\| T(s_0) - \frac{1}{n} g(s_0) \mathbb{1}\mathbb{1}^T \right\| = 0$$

Can this be extended to the heterogeneous case?

Coherence in networked dynamical systems: Heterogeneous case

For fixed s_0,

$$T(s_0) = V\left(V^T \mathrm{diag}\{g_i^{-1}(s_0)\}V + \Lambda\right)^{-1}V^T$$

$$= V\begin{bmatrix} \frac{\mathbb{1}^T}{\sqrt{n}}\mathrm{diag}\{g_i^{-1}(s_0)\}\frac{\mathbb{1}}{\sqrt{n}} & \frac{\mathbb{1}^T}{\sqrt{n}}\mathrm{diag}\{g_i^{-1}(s_0)\}V_\perp \\ V_\perp^T\mathrm{diag}\{g_i^{-1}(s_0)\}\frac{\mathbb{1}}{\sqrt{n}} & V_\perp^T\mathrm{diag}\{g_i^{-1}(s_0)\}V_\perp + \tilde{\Lambda} \end{bmatrix}^{-1} V^T$$

$$\frac{1}{n}\sum_{i=1}^n g_i^{-1}(s_0)$$

$$\tilde{\Lambda} = \mathrm{diag}\{\lambda_2(L), \cdots, \lambda_n(L)\}$$

$\lambda_2(L) \to \infty$

$$V\begin{bmatrix} \frac{1}{n}\sum_{i=1}^n g_i^{-1}(s_0) & \frac{\mathbb{1}_n^T}{\sqrt{n}}\mathrm{diag}\{g_i^{-1}(s_0)\}V_\perp \\ V_\perp^T\mathrm{diag}\{g_i^{-1}(s_0)\}\frac{\mathbb{1}_n}{\sqrt{n}} & V_\perp^T\mathrm{diag}\{g_i^{-1}(s_0)\}V_\perp + \tilde{\Lambda} \end{bmatrix}^{-1} V^T \xrightarrow{\lambda_2(L)\to\infty} V\begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^n g_i^{-1}(s_0)\right)^{-1} & \mathbb{0}_{1\times(n-1)} \\ \mathbb{0}_{(n-1)\times 1} & \mathbb{0}_{(n-1)\times(n-1)} \end{bmatrix} V^T = \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^n g_i^{-1}(s_0)\right)^{-1}\mathbb{1}\mathbb{1}^T$$

$$:= \bar{g}(s_0)$$

The **minimum singular value** grows unbounded as $\lambda_2(L)$ increases

The coherent dynamics: **Harmonic mean** of all $g_i(s)$

Coherence in networked dynamical systems: Heterogeneous case

Theorem (Coherence as the pointwise convergence of transfer matrix). *Define* $\bar{g}(s) = \left(\frac{1}{n}\sum_{i=1}^{n} g_i^{-1}(s)\right)^{-1}$. *If* $s_0 \in \mathbb{C}$ *is neither a zero nor a pole of* $\bar{g}(s)$, *then we have*

$$\lim_{\lambda_2(L)\to+\infty} \left\| T(s_0) - \frac{1}{n}\bar{g}(s_0)\mathbb{1}\mathbb{1}^T \right\| = 0\,.$$

- We can further prove **uniform convergence** over a compact subset of complex plane, if it doesn't contain any zero nor pole of $\bar{g}(s)$
- Convergence in transfer matrix is **related to time-domain response** by Inverse Laplace Transform
- Algebraic connectivity of L is an indicator of **level of coherence**

This Talk

- Characterization of **coherent dynamics**
- **Dynamics concentration** of large-scale tightly-connected network
- Numerical illustrations

Dynamics Concentration: From deterministic to stochastic

The coherent dynamics of the network is given by

$$\bar{g}(s) = \left(\frac{1}{n}\sum_{i=1}^{n} g_i^{-1}(s)\right)^{-1}$$

Suppose $g_i(s)$ are "i.i.d. random transfer functions", then for fixed s_0, $\bar{g}(s_0)$ is the harmonic mean of complex random variables

It converges in probability to a **deterministic** value as network size n increases

The "expected "coherent dynamics is given by

$$\hat{g}(s) = \left(\mathbb{E}Re(g_i^{-1}(s)) + j\mathbb{E}Im(g_i^{-1}(s))\right)^{-1} := \left(\mathbb{E}g_i^{-1}(s)\right)^{-1}$$

If we let the **network size n grows**, and in the meantime, **increase the network connectivity,** we would expect that for fixed s_0

$$T(s_0) \xrightarrow{\mathcal{P}} \frac{1}{n}\hat{g}(s_0)\mathbb{1}\mathbb{1}^T$$

Dynamics Concentration: Stochastic convergence result

Theorem (Coherence as the pointwise convergence of transfer matrix). *Define* $\bar{q}(s) = \left(\frac{1}{n}\sum_{i=1}^{n} g_i^{-1}(s)\right)^{-1}$. *If* $s_0 \in \mathbb{C}$ *is neither a zero nor a pole of* $\bar{q}(s)$, *then*

Definition. *A random variable* X *is a sub-Gaussian random variable if* $\forall t > 0$:

$$\mathbb{P}(|X| \geq t) \leq 2\exp\left(-ct^2\right) ,$$

for some $c > 0$.

Theorem (Dynamics Concentration). *Con... ...rks under graph Laplacian* L *with algebraic connectivity satisfying* $\lambda_2(L$ ... $)$ *for some* $p \in (0, 1]$. *Let* $g_i(s)$ *be i.i.d. random transfer functions. Given* s_0 ..., *suppose that* $g_i^{-1}(s_0)$ *has both its real and imaginary part given by sub-Gaussian random variables, and* s_0 *is not a pole of* $\hat{g}(s) = \left(\mathbb{E}g_i^{-1}(s)\right)^{-1}$. *Then* $\forall \epsilon > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\left\| T(s_0) - \frac{1}{n}\hat{g}(s_0)\mathbb{1}\mathbb{1}^T \right\| \geq \epsilon\right) = 0$$

Dynamics Concentration: Stochastic convergence result

Theorem (Dynamics Concentration). *Consider the networks under graph Laplacian L with algebraic connectivity satisfying $\lambda_2(L) = \Omega(n^p)$ for some $p \in (0,1]$. Let $g_i(s)$ be i.i.d. random transfer functions. Given $s_0 \in \mathbb{C}$, suppose that $g_i^{-1}(s_0)$ has both its real and imaginary part given by sub-Gaussian random variables, and s_0 is not a pole of $\hat{g}(s) = \left(\mathbb{E} g_i^{-1}(s)\right)^{-1}$. Then $\forall \epsilon > 0$,*

$$\lim_{n \to +\infty} \mathbb{P}\left(\left\| T(s_0) - \frac{1}{n}\hat{g}(s_0)\mathbb{1}\mathbb{1}^T \right\| \geq \epsilon \right) = 0$$

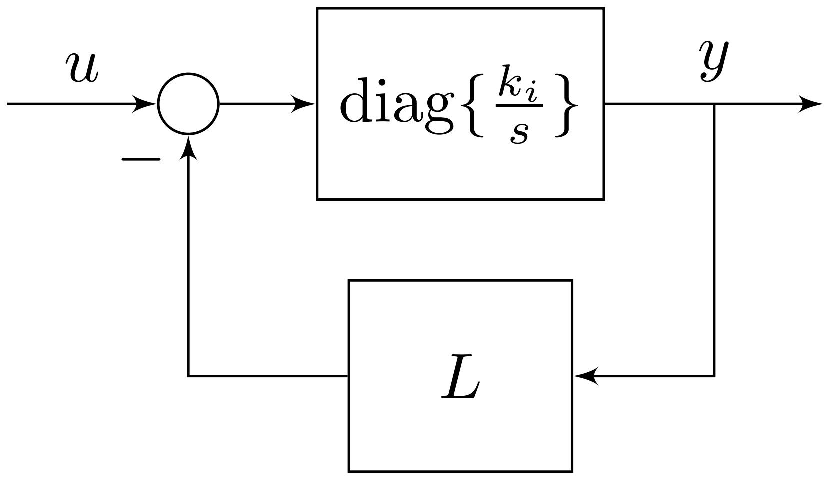
- Tightly-connected networks exhibit strong coherence and the coherent dynamics is given by the harmonic mean of all node dynamics
- The coherent dynamics of a stochastic network converges to a deterministic dynamics as network size grows

This Talk

- Characterization of **coherent dynamics**
- **Dynamics concentration** of large-scale tightly-connected network
- Numerical illustrations

Numerical illustration: A different view of Consensus network

Consider a first-order consensus network, where each node has different "acceptance rate":



Impulse response of this network gives exactly the evolution of opinions **y** starting from an initial opinions $\mathbf{y_0}$

Numerical illustration: A different view of Consensus network

Frequency Domain **(Coherence and Dynamics Concentration)**	Time Domain (State-space Analysis)

Numerical illustration: Consensus on tightly-connected networks

Simulation settings:

- Random nodal acceptance rate:

$$k_i \overset{i.i.d.}{\sim} Unif[1,5]$$

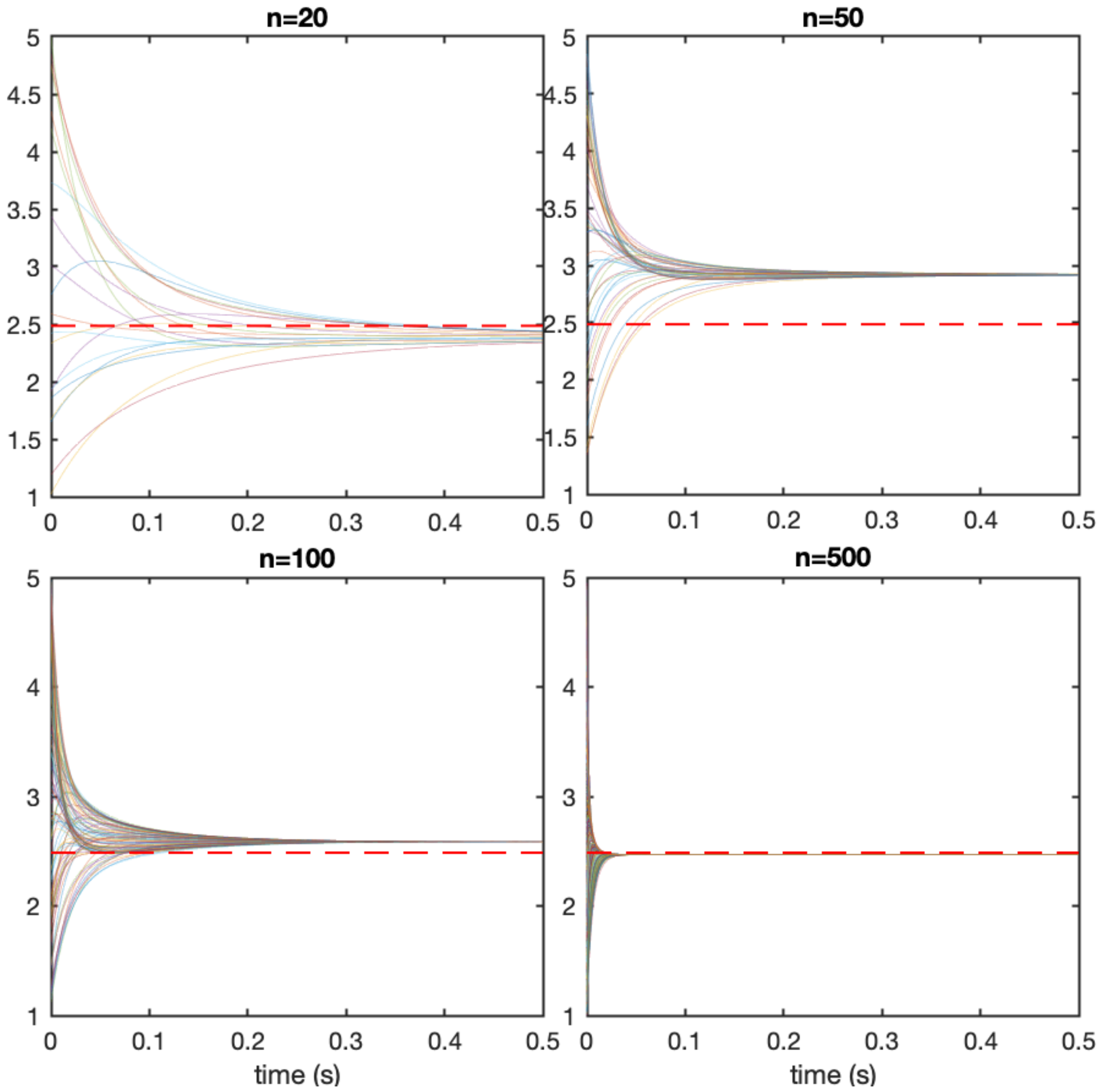
- Tightly-connected graph: L is the Laplacian of d-regular ring, d is roughly n/3

The ”expected” coherent dynamics:

$$\hat{g}(s) = \left(\mathbb{E}k_i^{-1}\right)^{-1}\frac{1}{s} = \frac{4}{\ln 5}\frac{1}{s}$$

its impulse response is shown in dashed red line

The coherent dynamics accurately represents the entire network as the consequence of Dynamics Concentration



Conclusion

- We proved that tightly-connected networks exhibit strong coherence, and the coherent dynamics is given by the harmonic mean of all node dynamics
- In a stochastic network where node dynamics are represented by i.i.d. random transfer functions, we showed that the coherent dynamics of the network converges to a deterministic one as the network size grows
- In numerical illustration, we provided the case where the consensus network exhibits Dynamics Concentration

Thank you for your attention!